Onset of instability for the Alber equation, and applications for rogue waves

Agissilaos G. Athanassoulis

Department of Mathematics, University of Dundee

Joint work with I. Kyza (St Andrews) & F. Karakatsani (U. of Ioannina)

AIMS 2026 – Special Session 51

Recent progress on rogue waves and their applications

This talk

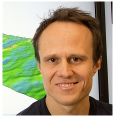
Irene Kyza St Andrews

Fotini Karakatsani Ioannina

Mariya Ptashnyk Heriot-Watt

Gerassimos Athanassoulis NTUA

Themistoklis Sapsis MIT



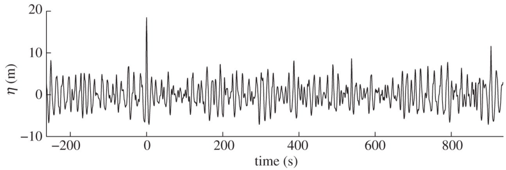
Odin Gramstad DNV

- A. Athanassoulis, G. Athanassoulis and T. Sapsis, **Journal of Ocean Engineering and Marine Energy** (2017)
- A. Athanassoulis, G. Athanassoulis, M. Ptashnyk and T. Sapsis, **Kinetic and Related Models** (2020)
- A. Athanassoulis and O. Gramstad, **Fluids** (2021)
- A. Athanassoulis and I. Kyza, **Water Waves** (2023)
- A. Athanassoulis, I. Kyza and F. Karakatsani, to appear in **Applied Numerical Mathematics** (arXiv:2506.06879)
- A. Athanassoulis, **arXiv:2604.16998**

Two main meanings of "Rogue waves" in this session

(A) Oceanography

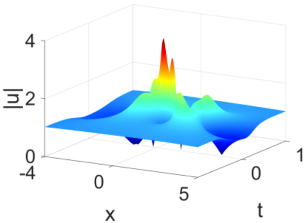
- A **localized event** far larger than the RMS size in a stochastic wave field.
- Observed in nature, in wavefields that are characterized through their **power spectrum** (**not explicitly known**).
- **Modulation instability** is relevant.



Draupner platform, the North Sea

(B) Integrable equations

- A solution of a nonlinear dispersive equation with **sharply localized features**, on a background.
- The background is **explicitly known** (plane wave; more recently elliptic / periodic etc).
- **Modulation instability** is relevant.



Sasa-Satsuma RW [Wu et al.]

Deriving the Alber equation (sketch)

1. Envelope. A envelope for unidirectional sea state $\to$ focusing cubic NLS:

$$i\partial_t A + p\,\Delta A + \tfrac{q}{2}|A|^2 A = 0.$$

2. Stochastic modeling: Second moments. Form two-space autocorrelation $R(x,y,t) = \mathrm{E}\Big[A(x,t)\overline{A(y,t)}\Big]$. A Gaussian (BBGKY) closure gives, to leading order,

$$i\partial_t R + p(\Delta_x - \Delta_y)R + q\big(R(x,x,t) - R(y,y,t)\big)R = 0. \quad \text{[Alber 1978]}$$

3. Constitutive assumption: quasi-homogeneity. Seas are nearly homogeneous, so split

$$R(x,y,t) = \underbrace{\Gamma(x-y)}_{\text{known background}} + \underbrace{u(x,y,t)}_{\text{inhomogeneity}}$$

$$\boxed{\begin{gathered} i\partial_t u + p(\Delta_x - \Delta_y)u + q\big(V(x,t) - V(y,t)\big)\big(\Gamma(x-y) + u\big) = 0 \\ V(x,t) = u(x,x,t) \end{gathered}}$$

- Γ (equivalently the power spectrum $P = \mathcal{F}\Gamma$) is **data measured from the field**.
- Equation structure related to a *von Neumann equation with δ self-interaction* / mixed-state NLS.

The difficulties

The Alber equation

$$i\partial_t u + p(\Delta_x - \Delta_y)u + q\big(u(x,x,t) - u(y,y,t)\big)\big(\Gamma(x-y) + u\big) = 0$$

is a dispersive nonlinear equation. But:

- Hyperbolic Laplacian not sign-definite $\Rightarrow$ **lose energy bounds**
- trace operator $u(x,y,t) \mapsto V(x,t) = u(x,x,t) \Rightarrow$ **lose $\frac{d}{2}$ derivatives**
- $\partial_t u(x,y,t)$ depends on $u(x,y,t)$, $u(x,x,t)$, $u(y,y,t) \Rightarrow$ **non-locality**

The bifurcation

The Alber equation can be re-written as

$$i\partial_t u + p(\Delta_x - \Delta_y)u + q\big(u(x,x,t) - u(y,y,t)\big)\Gamma(x-y) = q\big(u(y,y,t) - u(x,x,t)\big)u$$

$$i\partial_t u + \mathcal{L}_\Gamma u = \mathcal{Q}(u,u)$$

- The linear problem undergoes a bifurcation depending solely on Γ [Alber1978; A. et al. 2020]. $\Rightarrow$ **MI may be ON or OFF!**
- Does fully nonlinear stability follow the linear stability analysis? **No spectral gap $\Rightarrow$ not automatic**
- What happens in the fully nonlinear problem in in the unstable regime? $\Rightarrow$ **ramifications for Oceanic Rogue Waves**

Global well-posedness

Operator-valued (mixed-state) form:

$$i\partial_t\gamma + p\,[\Delta,\gamma] + q\,[V_{\rho_\gamma},\gamma] = 0, \qquad \rho_\gamma(x,t) = K_\gamma(x,x,t),$$

with $\gamma = \gamma^* \geq 0$ trace-class, kernel $K_\gamma(x,y,t) = \Gamma(x-y) + u(x,y,t)$, and V_f the operator of multiplication by f.

Theorem (GWP in $H^1\mathfrak{S}^1(\mathbb{T})$; both signs, no smallness [A. arXiv:2604.16998])

Let $pq \neq 0$ and $\gamma_0 \in H^1\mathfrak{S}^1(\mathbb{T})$ be self-adjoint and ***non-negative****. Then the Alber equation has a unique global solution $\gamma \in C(\mathbb{R}; H^1\mathfrak{S}^1(\mathbb{T}))$, conserving mass $\|\gamma\|_{\mathfrak{S}^1}$, $\|\gamma\|_{\mathfrak{S}^2}$ and energy $E(\gamma)$, with a uniform bound $\|\gamma(t)\|_{H^1\mathfrak{S}^1} \leq \bar{Y}$ for all t. Higher regularity $H^s\mathfrak{S}^1$ ($s \geq 2$) propagates.*

Ingredients (high level):

- *Schatten–Sobolev spaces* $H^s\mathfrak{S}^1$: control the position density ρ_γ **without losing derivatives**.
- **Energy conservation** + *Hoffmann–Ostenhof* + *Gagliardo–Nirenberg*:

$$\|\rho_\gamma\|_{L^2}^2 \leq C\,\|\gamma\|_{\mathfrak{S}^1}^{3/2}\sqrt{\operatorname{tr}(-\Delta\gamma(t))} + C\,\|\gamma\|_{\mathfrak{S}^1}^2 \implies \operatorname{tr}(-\Delta\gamma(t)) \quad \leq A + B\sqrt{\operatorname{tr}(-\Delta\gamma(t))}$$

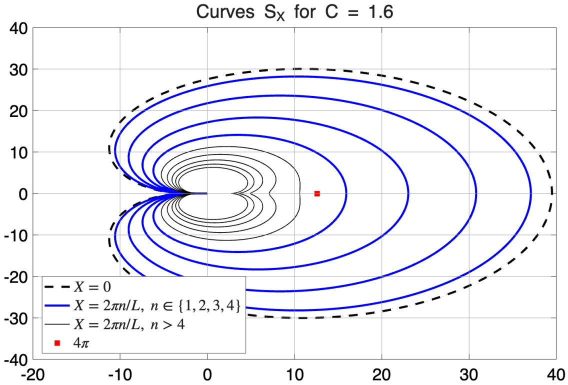
- Non-negativity ($\gamma \geq 0$) is a *structural* requirement (roughly reflecting the non-negativity of the background power-spectrum $P = \mathcal{F}\Gamma$).

Linear stability: the Penrose condition

The wavenumber X_* is unstable iff

$$\exists\, \omega_*,\ \mathrm{Re}\ \omega_* > 0: \qquad \mathbb{H}\left[\frac{P\left(k+\frac{X_*}{2}\right) - P\left(k-\frac{X_*}{2}\right)}{X_*}\right](\omega_*) \;=\; \frac{4\pi p}{q},$$

where $\mathbb{H}$ is the Hilbert transform (acting in k) and $P = \mathcal{F}\Gamma$ the power spectrum.



Instability is read off the background spectrum $P = \mathcal{F}\Gamma$:

$$S_X(t) = \mathbb{H}[D_X P](t) - i\, D_X P(t)$$

- Wavenumber X is **unstable** $\iff$ S_X winds around $4\pi p/q$.
- *Baseband*: only $|X| < X_{\max}$ unstable.
- $2\pi/X_{\max}$ $\to$ lengthscale of the coherent structure
- $\max \mathrm{Re}\, \omega_*$ $\to$ rate of growth.

Why a new numerical method?

- Despite its standing in oceanography, the Alber equation had **one** bespoke numerical method (Stiassnie–Regev–Agnon 2008; Ribal et al. 2013):
  forward Euler + 2nd-order FD + a non-standard far-field boundary condition fitted to each spectrum.
- We aim for structure-preserving; 2^{nd}-order in time + high order in space; easy to implement

Idea

- Frame the Alber equation as an NLS variant on $\mathbb{T}$

$$i\partial_t u + p(\Delta_x - \Delta_y)u + q\big(u(x,x,t) - u(y,y,t)\big)\big(\Gamma(x-y) + u\big) = 0$$

- Use a spatial discretization that allows "silent trace" (spectral / FD on symmetric grid)
- Devise a Besse-type scheme (relaxation Crank-Nicolson) [Besse 2004; Besse et al. 2021]

  $\Rightarrow 2^{nd}$ **order in time** $\Rightarrow$ **linearly implicit** $\Rightarrow$ **structure preserving**

The scheme [A., Kyza & Karakatsani, APNUM 2026]

Auxiliary variable $\phi = u(x,x) - u(y,y)$ on a *staggered* time grid:

$$\frac{1}{2}\Big(\Phi^{n+\frac{1}{2}} + \Phi^{n-\frac{1}{2}}\Big) = U^n(x,x) - U^n(y,y),$$

$$i\,\frac{U^{n+1} - U^n}{\tau} + p(\Delta_x - \Delta_y)U^{n+\frac{1}{2}} + q\,\Phi^{n+\frac{1}{2}}\big(\Gamma + U^{n+\frac{1}{2}}\big) = 0.$$

- 2nd order in time, **linearly implicit** (explicit in the nonlinearity).
- 4th-order finite differences in space.
- **Structure-preserving**: discrete L^2 *balance law*
- Two invariants conserved at the discrete level.

Balance Law carries over to the discrete level:

Continuous $\|u\|_{L^2}$ Balance Law $\frac{d}{dt}\|u\|_{L^2}^2 = 2\,\mathrm{Re}\big[iq\,\big\langle (V(x){-}V(y))\,\Gamma(x{-}y),\, u\big\rangle\big]$

Discrete $\|U^n\|_{L^2}$ Balance Law $\frac{\|U^{n+1}\|_{L^2}^2 - \|U^n\|_{L^2}^2}{\tau} = 2\,\mathrm{Re}\big[iq\,\big\langle \Phi^{n+\frac{1}{2}}\,\Gamma(x{-}y),\, U^{n+\frac{1}{2}}\big\rangle\big]$

Implementation

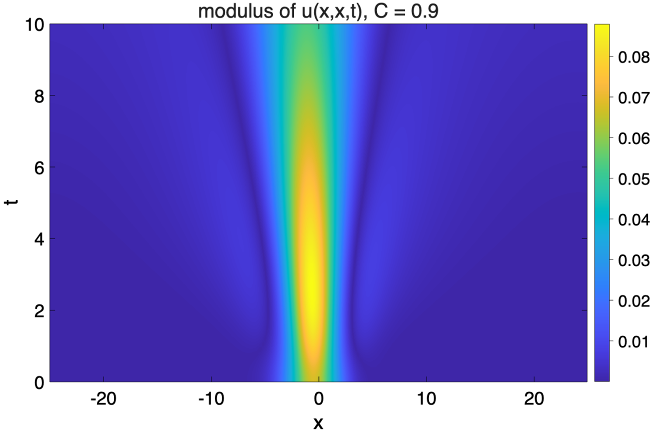
- Uniform $N \times N$ grid on $[-\frac{L}{2}, \frac{L}{2}]^2$, mesh size $h = L/N$
- arrays $U_{i,j}^n$ approximating the solution at (x_i, y_j, t^n) and $\Phi_{i,j}^{n+\frac{1}{2}}$ at the nodes $(x_i, y_j, t^{n+\frac{1}{2}})$.
- $\mathtt{vec}(\cdot)$ is canonical vectorization of an $N \times N$ array into a length-N^2 vector
- $\mathfrak{D}_H$: sparse $N^2 \times N^2$ matrix for the hyperbolic Laplacian with periodic BCs
- $\mathfrak{F}^{n+\frac{1}{2}} = \mathtt{diag}\big(\mathtt{vec}(\Phi_{i,j}^{n+\frac{1}{2}})\big)$: pointwise product by $\Phi^{n+\frac{1}{2}}$; I: identity.

Each step: update Φ, solve one sparse linear system for $U^{n+\frac{1}{2}}$, extract U^{n+1}:

$$\boxed{\begin{aligned}
&\Phi_{i,j}^{n+\frac{1}{2}} = 2\big(U_{i,i}^n - U_{j,j}^n\big) - \Phi_{i,j}^{n-\frac{1}{2}},\\
&\Big(I - i\tfrac{p\tau}{2}\,\mathfrak{D}_H - i\tfrac{q\tau}{2}\,\mathfrak{F}^{n+\frac{1}{2}}\Big)\,\mathtt{vec}(U_{i,j}^{n+\frac{1}{2}}) = \mathtt{vec}\big(U_{i,j}^n + i\tfrac{q\tau}{2}\,\Gamma(x_i - y_j)\,\Phi_{i,j}^{n+\frac{1}{2}}\big),\\
&U_{i,j}^{n+1} = 2U_{i,j}^{n+\frac{1}{2}} - U_{i,j}^n.
\end{aligned}}$$

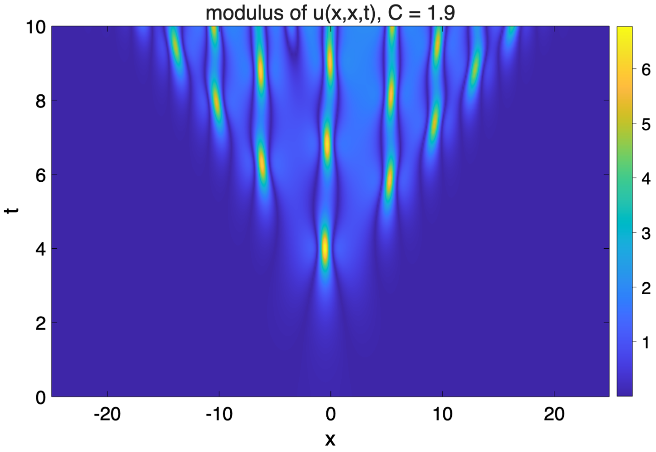
Structure: $M = I - i\frac{\tau}{2}(p\,\mathfrak{D}_H + q\,\mathfrak{F}^{n+\frac{1}{2}}) \Rightarrow M$ normal, invertible $\forall\,\tau$, $\|M^{-1}\|_2 \le 1$.

Stable regime: nonlinear stability



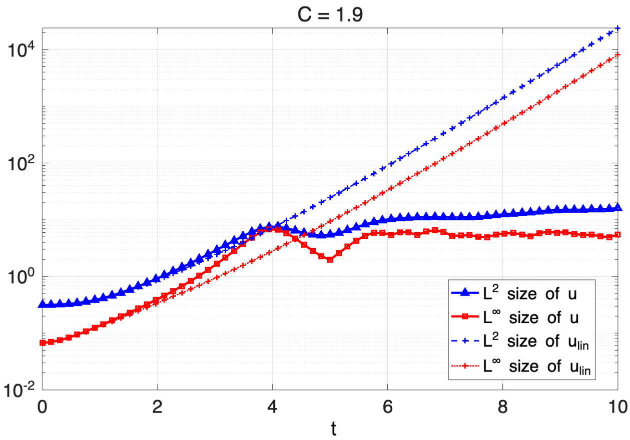
- $|u(x, x, t)|$ for a Penrose-stable spectrum
- No meaningful growth in time – the inhomogeneity simply **disperses**.
- The background Γ would naively act as a source, yet u stays small.
- Proving smallness for all time on $\mathbb{T}$ is open.

Unstable regime: the space–time lattice



- $|u(x,x,t)|$ for a (strongly) Penrose-unstable spectrum
- Stage 1: exponential growth to a single localised coherent structure, $O(1)$, peaking ≈ 6 (initially $O(10^{-2})$).
- Stage 2: **no further L^∞ growth**; a lattice of similar structures spreads outward in a space–time cone.
- *Attracting pattern*: eyeball-robust to the initial u_0, depends on p, q, Γ.
- Long-time limit open; on $\mathbb{T}$ expected to be different than $\mathbb{R}$.

Linear theory: right about what it can see

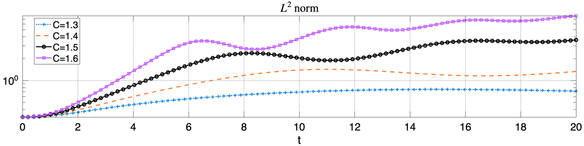


Norms of the full (u) vs. linearised (u_{lin}) solution for a Penrose-unstable problem.

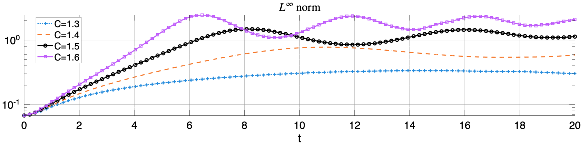
- Linear stability analysis nails the **onset** and the **initial growth rate**.
- It completely **misses** the maximum amplitude, the coherent structure, and the relevant timescales.

$\Rightarrow$ The nonlinear problem must be solved to say anything about extremes.

The bifurcation is gradual – not a switch at onset



L^2 growth vs. time, several C.



L^∞ growth vs. time, several C.

- $C \approx 1$ is the critical point for the bifurcation.
- Linear instability is present for $C > 1$, but for *intermediate* intensities the maxima **plateau** at values negligible next to the original background.
- **Large growth of u compared to u_0** $\neq$ **events that dominate Γ**
- For a physically relevant extreme event to actually form, *sufficiently strong* instability is required.

Two amplification factors – and why the distinction matters

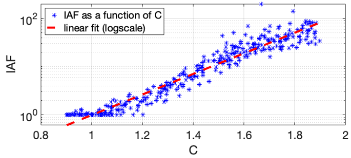
$$\mathrm{IAF} = \frac{\max_{t,x,y} |u|}{\max_{x,y} |u_0|}$$

$$\mathrm{TAF} = \frac{\max_{t,x,y} |u + \Gamma|}{\Gamma(0)}$$

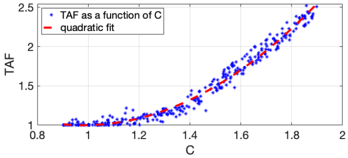
- **IAF**: how much the inhomogeneity grows (instability strength; numerical difficulty).
- Large IAF is **necessary but not sufficient** for large TAF.

- **TAF**: how large the sea state gets *relative to the background* – the physical quantity.
- Oceanic Rogue Wave guaranteed for $\mathrm{TAF} > 2.84$ [Ribal et al., JFM, 2013]

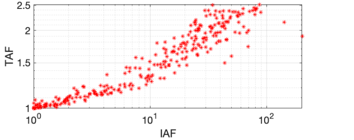
Monte Carlo: the background decides, not the seed



IAF vs. C



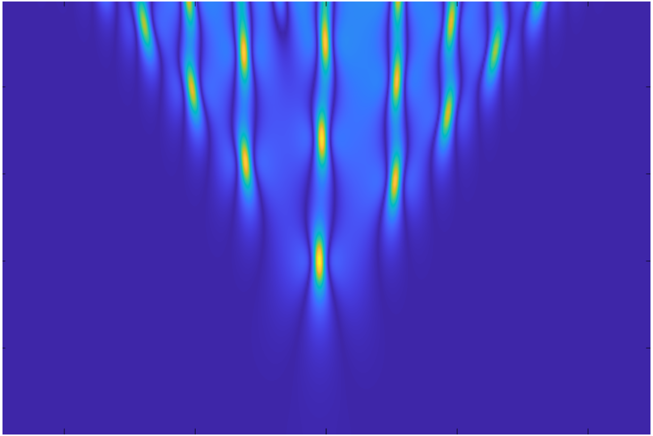
TAF vs. C



TAF vs. IAF

- 310 randomised initial conditions, $C \sim \mathcal{U}[0.9, 1.9]$.
- Amplification factors depend essentially **only on the background** Γ (i.e. on C), *not* on the initial inhomogeneity.
- $\mathrm{TAF}(C)$ grows *quadratically*: **no jump** at onset $C \approx 1$; a substantial effect ($\mathrm{TAF} \approx 1.4$, i.e. $+40\%$ variance) only from $C \approx 1.35$ or larger.
- **Physically plausible values:** $\mathrm{TAF} \approx 1.5$, much smaller than 2.84.

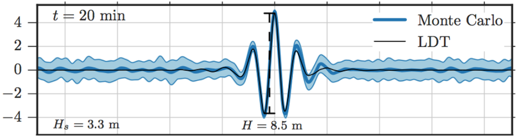
Ramifications for rogue waves



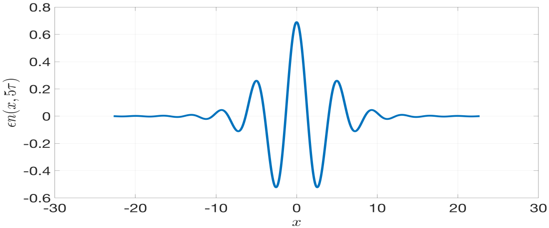
- Lattice of **hotspots of increased variance of the wavefield** found.
- The variance is not strong enough to guarantee Oceanic Rogue Waves, but it is strong enough to make them orders of magnitude more likely.

- Each peak is a "rogue wave" in a new nonlinear dispersive equation.
- Novel feature: it depends on Γ.
- Open questions: can we find an expression for it? Can we predict the structure of the lattice?

Conclusions



"Universal profile of rogue waves"
[van den Eijden et al. PNAS, 2018]



Profile predicted by the growth of Penrose-unstable modes
[A. et al., JOEME, 2017]

- The Alber equation presents a bifurcation that is only now being understood.
- It has clear applicability to oceanic rogue waves.
- Moreover, through generalized modulation instability, it generates new solutions with sharply localized features.

Thank you!

Appendix: initialization of the auxiliary variable

The first step needs $\Phi_{i,j}^{-\frac{1}{2}}$. Two choices:

$$\textbf{Naive:} \quad \Phi_{i,j}^{-\frac{1}{2}} = U_{i,i}^{0} - U_{j,j}^{0}.$$

$$\textbf{Advanced:} \quad \begin{cases} \Phi_{i,j}^{*} = U_{i,i}^{0} - U_{j,j}^{0}, \\ \left(I + i\frac{p\tau}{4}\mathfrak{D}_H + i\frac{q\tau}{4}\mathfrak{F}^*\right)\mathtt{vec}(U_{i,j}^{-\frac{1}{4}}) = \mathtt{vec}\big(U_{i,j}^{0} - i\frac{q\tau}{4}\Gamma(x_i - y_j)\Phi_{i,j}^{*}\big), \\ U_{i,j}^{-\frac{1}{2}} = 2U_{i,j}^{-\frac{1}{4}} - U_{i,j}^{0}, \qquad \Phi_{i,j}^{-\frac{1}{2}} = U_{i,i}^{-\frac{1}{2}} - U_{j,j}^{-\frac{1}{2}}. \end{cases}$$

- **Naive** (Besse): 2^{nd} order in u, but only 1^{st} **order in time** for ϕ.
- **Advanced** (half-step predictor): full order in *both* u and ϕ.
- The main variable u is largely insensitive to the error in ϕ (an averaging effect).

Appendix: experimental order of convergence

Time refinement ($h = 0.04$ fixed)

naive ($\mathrm{EOC}_\phi \approx 1$):

τ	$\mathcal{E}_u$	EOC_u	$\mathcal{E}_\phi$	EOC_ϕ
3.0e−2	5.3e−3	–	9.6e−2	–
2.1e−2	2.7e−3	1.96	6.6e−2	1.06
1.5e−2	1.3e−3	2.05	4.6e−2	1.04
1.1e−2	6.7e−4	1.99	3.2e−2	1.03

advanced:

τ	$\mathcal{E}_u$	EOC_u	$\mathcal{E}_\phi$	EOC_ϕ
3.0e−2	4.6e−3	–	1.1e−2	–
2.1e−2	2.3e−3	1.96	5.4e−3	2.01
1.5e−2	1.1e−3	2.06	2.7e−3	1.99
1.1e−2	5.7e−4	1.98	1.3e−3	2.01

Space refinement ($\tau = 5 \cdot 10^{-4}$ fixed)

naive:

h	$\mathcal{E}_u$	EOC_u	$\mathcal{E}_\phi$	EOC_ϕ
0.40	1.3e−2	–	2.4e−2	–
0.34	6.7e−3	3.89	1.2e−2	3.78
0.28	3.3e−3	4.07	6.3e−3	3.88
0.24	1.7e−3	4.04	3.4e−3	3.55

advanced:

h	$\mathcal{E}_u$	EOC_u	$\mathcal{E}_\phi$	EOC_ϕ
0.40	1.3e−2	–	2.3e−2	–
0.34	6.7e−3	3.89	1.2e−2	3.86
0.28	3.3e−3	4.07	5.9e−3	4.06
0.24	1.7e−3	4.04	2.9e−3	4.03

Appendix: the a priori $H^1\mathfrak{S}^1$ bound (no smallness)

Kinetic energy $\mathcal{K}(t) := \mathrm{tr}(-\Delta\gamma(t))$; conserved energy and mass

$$E(\gamma) = p\ \mathrm{tr}(\Delta\gamma) + \tfrac{q}{2}\|\rho_\gamma\|_{L^2}^2, \qquad \mathcal{M} := \|\gamma\|_{\mathfrak{S}^1} = \|\rho_\gamma\|_{L^1}\ (\gamma \geq 0).$$

Focusing ($pq > 0$). Energy conservation $\Rightarrow |p|\,\mathcal{K}(t) \leq |E(\gamma_0)| + \frac{|q|}{2}\|\rho_\gamma\|_{L^2}^2$, so it is enough to bound $\|\rho_\gamma\|_{L^2}^2$ by $\mathcal{K}$. Two inequalities (the first *needs* $\gamma \geq 0$):

$$\underbrace{\int_{\mathbb{T}} |\nabla\sqrt{\rho_\gamma}|^2 \leq \mathrm{tr}(-\Delta\gamma),}_{\text{Hoffmann–Ostenhof}} \qquad \underbrace{\|f\|_{L^4}^4 \leq \tfrac{1}{|\mathbb{T}|}\|f\|_{L^2}^4 + 2\|f\|_{L^2}^3\|\nabla f\|_{L^2}}_{\text{Gagliardo–Nirenberg on } \mathbb{T}}.$$

Take $f = \sqrt{\rho_\gamma}$, use $\|\sqrt{\rho_\gamma}\|_{L^2}^2 = \|\rho_\gamma\|_{L^1} = \mathcal{M}$:

$$\|\rho_\gamma\|_{L^2}^2 = \|\sqrt{\rho_\gamma}\|_{L^4}^4 \ \leq\ C\,\mathcal{M}^{3/2}\sqrt{\mathcal{K}(t)} + C\,\frac{\mathcal{M}^2}{|\mathbb{T}|} \qquad (\textbf{sublinear in } \mathcal{K}).$$

Feed back into the energy inequality:

$$\mathcal{K}(t)\ \leq\ A + B\sqrt{\mathcal{K}(t)} \quad \Longrightarrow \quad \mathcal{K}(t) \leq \bar{Y}\ \ \forall t \quad (\textbf{no smallness}).$$

Defocusing ($pq < 0$). Kinetic and potential terms share a sign, so $|p|\,\mathcal{K}(t) \leq |E(\gamma_0)|$ *directly*.

Towards nonlinear stability

Take a homogeneous background $\Gamma = \Gamma^* \geq 0$ that is **Penrose-stable** (linearly stable: no unstable modes, with a stability margin $\kappa > 0$). Perturb it by a small, self-adjoint U_0 with $\|\boldsymbol{U_0}\|_{\boldsymbol{H^1\mathfrak{S}^1}} \leq \boldsymbol{\varepsilon}$ and $\gamma_0 = \Gamma + U_0 \geq 0$.

Theorem (Polynomial-in-t control near Penrose-stable Γ [A. arXiv:2604.16998])

The solution $\gamma(t)$ exists globally in $C(\mathbb{R}; H^1\mathfrak{S}^1(\mathbb{T}))$ and

$$\|\gamma(t) - \Gamma\|_{H^1\mathfrak{S}^1} \;\leq\; 2\, C_\star(\Gamma)\, \langle t \rangle^2\, \varepsilon \qquad \text{for all } |t| \leq T_\star, \quad T_\star = c_\Gamma\, \varepsilon^{-1/5},$$

with $C_\star(\Gamma), c_\Gamma > 0$ independent of ε. In particular $\|\gamma(t) - \Gamma\|_{H^1\mathfrak{S}^1} \lesssim \varepsilon^{3/5}$ on that window.

- Bare well-posedness gives only the exponential $\|\gamma(t) - \Gamma\|_{H^1\mathfrak{S}^1} \leq \varepsilon\, e^{Ct}$; Penrose stability upgrades this to **polynomial** growth on the long window $|t| \lesssim \varepsilon^{-1/5}$.
- On Penrose-*unstable* backgrounds the exponential *is* attained ($\|\gamma(t) - \Gamma\| \gtrsim \varepsilon\, e^{ct}$) — this is the classical modulation instability.
- Falls short of genuine orbital stability ($\|\gamma(t) - \Gamma\| = o(1)$ as $t \to \infty$), which we conjecture but do not prove.